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Journal of Approximation Theory 164 (2012) 1427–1442
Journal of Approximation Theory
www.elsevier.com/locate/jat

Full length article

Null-series on the complex sphere

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Received 10 February 2012; received in revised form 1 June 2012; accepted 26 July 2012
Available online 6 August 2012

Communicated by Przemyslaw Wojtaszczyk

Abstract

We construct pluriharmonic null-series on the unit sphere of $\mathbb{C}^d$, $d \geq 2$. Also, we give examples when the corresponding spherical harmonics are sparsely distributed and have sufficiently small L^2-norms.

Keywords: Null-series; Pluriharmonic Riesz product; Spherical harmonic

1. Introduction

Let σ_d denote the normalized Lebesgue measure on the unit sphere $S_d = \{\zeta \in \mathbb{C}^d : |\zeta| = 1\}$. To distinguish the case $d = 1$, put $\mathbb{T} = S_1$ and $m = \sigma_1$. So, in what follows, the symbols S_d and σ_d are used for $d \geq 2$.

1.1. Null-series

By definition, a non-trivial series

$$\sum_{j\in\mathbb{Z}} c_j \zeta^j, \quad c_j \in \mathbb{C},\ \zeta \in \mathbb{T}, \tag{1.1}$$

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doi:10.1016/j.jat.2012.07.005

is called a trigonometric null-series if

$$\lim_{n\to\infty} \sum_{|j|\le n} c_j \zeta^j = 0 \quad \text{for } m\text{-almost all points } \zeta \in \mathbb{T}.$$

The first example of a trigonometric null-series was constructed by Menshov [9]. Generalizations of Menshov's result are usually related to series on locally compact non-discrete Abelian groups.

The sphere S_d is not a group; however, S_d is a homogeneous space. So, it is possible to develop the spectral function theory on S_d in terms of $H(p,q)$, the spaces of complex spherical harmonics. By definition, $H(p,q)$ is the space of homogeneous harmonic polynomials of bidegree $(p,q) \in \mathbb{Z}_+^2$ with respect to the variables $z_1, z_2, \dots, z_d$ and $\bar{z}_1, \bar{z}_2, \dots, \bar{z}_d$. In particular, $H(p,0)$ is the space of homogeneous holomorphic polynomials of degree p, $p \in \mathbb{Z}_+$. The spaces $H(p,q)$, $(p,q) \in \mathbb{Z}_+^2$, span $L^2(S_d) = L^2(S_d, \sigma_d)$ and are pairwise orthogonal. Thus, such standard notions of the classical harmonic analysis as that of the Fourier spectrum are applicable; see [11] for further details.

So, on the sphere S_d, it is natural to replace (1.1) by a non-trivial $H(p,q)$-series, $(p,q) \in \mathbb{Z}_+^2$. In this paper, we give examples of null-series with a sufficiently sparse $H(p,q)$-spectrum. Namely, we construct a pluriharmonic null-series, that is, a null-series of anti-holomorphic and holomorphic harmonics.

Theorem 1.1. *There exists a non-trivial sequence of holomorphic spherical harmonics $h_j \in H(j,0)$, $j \in \mathbb{Z}_+$, such that*

$$\lim_{n\to\infty} \sum_{j\le n} (\bar{h}_j + h_j)(\zeta) = 0 \quad \text{for } \sigma_d\text{-a.e. } \zeta \in S_d. \tag{1.2}$$

In fact, a somewhat stronger assertion holds. Let $E \subset \mathbb{Z}_+$. Assume that E contains arbitrarily long intervals. Then, in Theorem 1.1, we may additionally guarantee that $h_j = 0$ for all $j \in \mathbb{Z}_+ \setminus E$.

Applying Theorem 1.1 and identifying $\mathbb{C}^d$ and $\mathbb{R}^{2d}$, we obtain a series that consists of real spherical harmonics and converges to zero on the unit sphere of $\mathbb{R}^{2d}$. Such examples are probably of independent interest.

1.2. Slices of pluriharmonic null-series

Given $\zeta \in S_d$, the trigonometrical series

$$\sum_{j=0}^{\infty} (\bar{h}_j + h_j)(\lambda\zeta), \qquad \lambda \in \mathbb{T},$$

is called the slice of the series (1.2) at the point $\zeta \in S_d$. Theorem 1.1 and Fubini's theorem guarantee that a slice of the series (1.2) is a trigonometric null-series on $\mathbb{T}$ for σ_d-a.e. $\zeta \in S_d$. So, it is natural to ask whether it is possible to replace almost all slices by *all* slices (cf. [4], where an analogous problem is considered for the inner functions). In the present paper, we give a positive answer to this question.

Theorem 1.2. *There exists a non-trivial sequence of holomorphic spherical harmonics $h_j \in H(j,0)$, $j \in \mathbb{Z}_+$, such that the property*

$$\lim_{n\to\infty} \sum_{j\le n} (\bar{h}_j + h_j)(\lambda\zeta) = 0 \quad \text{for } m\text{-a.e. } \lambda \in \mathbb{T}$$

holds for all $\zeta \in S_d$.

Clearly, Theorem 1.2 implies Theorem 1.1. In fact, the proof of Theorem 1.1 will be used as a model for further arguments.

1.3. Trigonometric null-series with small coefficients

If $\sum_{j\in\mathbb{Z}} c_j\zeta^j$ is a null-series, then $\{c_j\} \notin \ell^2$. Therefore, if $b_j > 0$ and $\{b_j\} \notin \ell^2$, then it is natural to ask about the existence of a null-series with coefficients c_j such that $|c_j| \le b_j$. It is known that the answer is positive if the sequence b_j is sufficiently regular. In particular, by Ivašev-Musatov's theorem [5], the answer is positive for all standard sequences

$$b_j = \left(|j|\log|j|\log\log|j|\cdots\log_{(k)}|j|\right)^{-\frac{1}{2}}.$$

A trigonometric null-series is often constructed as the Fourier series of a singular (with respect to Lebesgue measure m) finite Borel measure μ such that $\widehat{\mu}(j) \to 0$ as $|j| \to \infty$; $m(K_\mu) = 0$, where K_μ is the closed support of μ. Indeed, the Riemannian theory guarantees that the Fourier series of μ converges to zero outside of K_μ. It is worth mentioning that similar methods do not provide pluriharmonic null-series on S_d. Indeed, let μ be a singular (with respect to σ_d) measure on S_d. If the Poisson integral $P[\mu]$ is a pluriharmonic function, then $\sigma_d(K_\mu) > 0$; moreover, it is known that the closed support of μ is the whole sphere S_d. Hence, the automatic convergence to zero is not guaranteed for the Fourier series of μ.

Körner [6] applied a different sophisticated construction to improve Ivašev-Musatov's theorem. Namely, an appropriate null-series exists if $b_{-j} = b_j$ for $j \ge 0$, the sequence b_j, $j \ge 0$, monotonically decreases and $\{b_j\} \notin \ell^2$.

Kozma and Olevskiĭ [7,8] also used a different method to construct trigonometric null-series with small anti-analytic parts.

Finally, note that Zygmund [13, Chapter 5, Theorem 7.7] constructed a trigonometric null-series as a Riesz product. This approach is the starting point for the arguments used in the present paper. More precisely, we apply pluriharmonic Riesz product constructions (see [1,3]).

1.4. Null-series of L^2-small spherical harmonics

By definition, a numerical sequence $\{b_j\}_{j=0}^{\infty}$, $b_j \ge 0$, is called L^2-admissible (more precisely, $L^2(S_d)$-admissible for $d \ge 2$) if there exists a non-trivial sequence of holomorphic spherical harmonics $h_j \in H(j, 0)$ such that property (1.2) holds and

$$\|h_j\|_{L^2(S_d)} \le b_j, \quad j = 0, 1, \ldots.$$

Note that $\{b_j\} \notin \ell^2$ if the sequence $\{b_j\}$ is L^2-admissible. Therefore, the following theorem is sharp, in a sense.

Theorem 1.3. *Let $b_0 = 1$ and let $b_j = j^{-\frac{1}{2}}$. Then the sequence $\{b_j\}$ is L^2-admissible.*

Related assertions about singular measures on the sphere S_d were obtained in [2].

Theorem 1.3 provides the required null-series on the sphere S_d for $d \ge 2$. Note that the proofs of similar quantitative results on the unit circle $\mathbb{T}$ are more sophisticated (cf. [5,6]).

2. A model: proof of Theorem 1.1

2.1. Auxiliary results

2.1.1. Lacunary Fourier series

Given a measure μ on $\mathbb{T}$ and a point $\lambda \in \mathbb{T}$, put

$$s_n[\mu](\lambda) = \sum_{j=-n}^{n} \widehat{\mu}(j)\lambda^j;$$

$$t_n[\mu](\lambda) = \frac{\widehat{\mu}(0)}{2} + \sum_{j=1}^{n} \widehat{\mu}(j)\lambda^j.$$

Let $(\lambda_1, \lambda_2) \subset \mathbb{T}$ denote the shortest arc joining the points $\lambda_1, \lambda_2 \in \mathbb{T}$. By definition, put

$$\mathcal{D}\mu(\lambda) = \lim_{|\theta|\to 0} \frac{\mu}{m}(\lambda, \lambda e^{i\theta}), \quad \lambda \in \mathbb{T},$$

provided that the above limit exists. Recall that $\mathcal{D}\mu(\lambda) = f(\lambda)$ for m-a.e. $\lambda \in \mathbb{T}$, where fm is the absolutely continuous part of the measure μ.

Lemma 2.1 (*See [13, Chapter 3, Theorems 8.1 and 1.27]*). *Let μ be a measure on the unit circle $\mathbb{T}$, $n_k \nearrow +\infty$ and $\widehat{\mu}(j) = 0$ for $n_k < |j| < 2n_k$, $k \in \mathbb{N}$. Then*
(i) $s_{n_k}[\mu](\lambda) \to \mathcal{D}\mu(\lambda)$,
(ii) *the sequence $t_{n_k}[\mu](\lambda)$ has a finite limit*
for m-a.e. $\lambda \in \mathbb{T}$.

2.1.2. Pluriharmonic projection

Put

$$PLH^2(S_d) = \operatorname{span}_{L^2(S_d)}\{H(p,q) : (p,q) \in \mathbb{Z}_+^2, \ pq = 0\}.$$

In other words, the space $PLH^2(S_d)$ consists of those functions $f \in L^2(S_d)$ for which the Poisson integral $P[f]$ is pluriharmonic. Let $\Pr : L^2(S_d) \to PLH^2(S_d)$ be the orthogonal projection.

Below we often consider polynomials $\Pr[fg]$, where $f \in H(p,0)$ and $g \in H(0,q)$. By [11, Theorem 12.4.4], we have

$$fg \in \sum_{j=0}^{\min(p,q)} H(p-j, q-j).$$

Therefore, $\Pr[fg] \in H(p-q, 0)$ for $p \geq q$ and $\Pr[fg] \in H(q-p, 0)$ for $p < q$.

2.1.3. Compact Hankel operators

Given a symbol $\varphi \in L^\infty(S_d)$, the formula

$$H_\varphi[f] = \varphi \Pr[f] - \Pr[\varphi f], \quad f \in L^2(S_d),$$

defines a Hankel-type operator. If φ is a polynomial, then $H_\varphi : C(S_d) \to C(S_d)$ is a compact operator, thus,

$$\left[\|f_j\|_{C(S_d)} \leq 1 \text{ and } f_j \to 0 \text{ weakly in } L^2(S_d)\right] \Rightarrow \|H_\varphi f_j\|_{C(S_d)} \to 0; \tag{2.1}$$

see [1] for further details.

2.1.4. Riesz pairs

Recall that Riesz [10] introduced the following product measures:

$$\prod_{k=1}^{\infty}\left(\frac{\overline{a}_k\overline{\zeta}^{j_k}}{2}+1+\frac{a_k\zeta^{j_k}}{2}\right),\quad \zeta\in\mathbb{T},\qquad |a_k|\le 1,\qquad \frac{j_{k+1}}{j_k}\ge 3.$$

To obtain appropriate analogs of the above products on the complex sphere, we replace the characters ζ^j, $j\in\mathbb{N}$, by suitable holomorphic homogeneous polynomials. Namely, Ryll and Wojtaszczyk [12] constructed holomorphic polynomials $R_j\in H(j,0)$, $j\in\mathbb{N}$, such that

- $\|R_j\|_{C(S_d)}=1$;
- $\|R_j\|_{L^2(S_d)}\ge\delta$ for a constant $\delta=\delta(d)>0$.

As in [1], we use the following brief notation: (R,a) is called a Riesz pair if $R=\{R_j\}_{j=1}^{\infty}$ is a Ryll–Wojtaszczyk sequence and $a=\{a_k\}_{k=1}^{\infty}\subset\mathbb{C}$, $0<|a_k|<1$.

2.2. *Proof of Theorem 1.1*

Let (R,a) be a Riesz pair such that $a\notin\ell^2$ and $a_k\to 0$. By induction, we construct an index sequence $J=\{j_k\}_{k=1}^{\infty}$, $j_{k+1}/j_k\ge 4$, a sign sequence $\beta=\{\beta_k\}_{k=1}^{\infty}$, $\beta_k\in\{\pm 1\}$, and an operator sequence $U=\{U_k\}_{k=1}^{\infty}\subset\mathcal{U}$. Here $\mathcal{U}=\mathcal{U}(d)$ denotes the group of unitary operators on the Hilbert space $\mathbb{C}^d$. A similar pluriharmonic Riesz product construction is used in the proof of Theorem 4.1(ii) from [1]; see [1] for further references.

2.2.1. Induction construction

Fix an index $j_1\in\mathbb{N}$ and put by definition $\varphi_1=1+\mathrm{Re}(a_1R_{j_1})$. Assume, as an induction hypothesis, that $k\in\mathbb{N}, \varphi_k>0$ on the sphere S_d, and the polynomial φ_k has the following homogeneous expansion:

$$\varphi_k=\sum_{p=n_1}^{n_k}\overline{h}_p+1+\sum_{p=n_1}^{n_k}h_p,\quad h_p\in H(p,0),\ n_k=j_1+\cdots+j_k. \tag{2.2}$$

Step $k+1$. We use an auxiliary index $\ell\in\mathbb{N}$ such that $\ell\ge 4j_k$.

Given positive functions $f,g\in L^1(S_d)$, we have

$$\int_{\mathcal{U}}\int_{S_d}f\cdot(g\circ U)\,d\sigma_d\,dU=\int_{S_d}f\,d\sigma_d\int_{S_d}g\,d\sigma_d,$$

where dU denotes the Haar measure on $\mathcal{U}$. Therefore, there exist operators $U_{k+1}^{\ell}\in\mathcal{U}$ such that

$$\int_{S_d}\varphi_k^{\frac{1}{2}}\left[\mathrm{Re}(a_{k+1}R_\ell\circ U_{k+1}^{\ell})\right]^2d\sigma_d\ge\int_{S_d}\varphi_k^{\frac{1}{2}}d\sigma_d\int_{S_d}\left[\mathrm{Re}(a_{k+1}R_\ell\circ U_{k+1}^{\ell})\right]^2d\sigma_d. \tag{2.3}$$

Since $a_{k+1}R_\ell\circ U_{k+1}^{\ell}$ is a non-constant homogeneous holomorphic polynomial, we have

$$\begin{aligned}\int_{S_d}\left[\mathrm{Re}\left(a_{k+1}R_\ell\circ U_{k+1}^{\ell}\right)\right]^2d\sigma_d&=\frac{1}{2}\int_{S_d}\left|a_{k+1}R_\ell\circ U_{k+1}^{\ell}\right|^2d\sigma_d\\&\ge\frac{\delta^2|a_{k+1}|^2}{2}.\end{aligned} \tag{2.4}$$

Note that $(1+x)^{\frac{1}{2}} + (1-x)^{\frac{1}{2}} \le 2(1-x^2/8)$ for $-1 \le x \le 1$, hence,

$$\int_{S_d} \left(\varphi_k^{\frac{1}{2}} [1+\mathrm{Re}(a_{k+1} R_\ell \circ U_{k+1}^\ell)]^{\frac{1}{2}} + \varphi_k^{\frac{1}{2}} [1-\mathrm{Re}(a_{k+1} R_\ell \circ U_{k+1}^\ell)]^{\frac{1}{2}} \right) d\sigma_d$$
$$\le 2 \int_{S_d} \varphi_k^{\frac{1}{2}} \left(1 - \frac{\left[\mathrm{Re}\left(a_{k+1} R_\ell \circ U_{k+1}^\ell\right)\right]^2}{8} \right) d\sigma_d$$
$$\le 2 \left(1 - \frac{\delta^2 |a_{k+1}|^2}{16} \right) \int_{S_d} \varphi_k^{\frac{1}{2}} \, d\sigma_d$$

by (2.3) and (2.4). Therefore, there exists a sign $\beta_{k+1}^\ell \in \{\pm 1\}$ such that

$$\int_{S_d} \varphi_k^{\frac{1}{2}} \left[1 + \mathrm{Re}\left(\beta_{k+1}^\ell a_{k+1} R_\ell \circ U_{k+1}^\ell \right) \right]^{\frac{1}{2}} d\sigma_d \le \left(1 - \frac{\delta^2 |a_{k+1}|^2}{16} \right) \int_{S_d} \varphi_k^{\frac{1}{2}} \, d\sigma_d.$$

Put $\varphi_{k+1}(\ell) = \Pr\left(\varphi_k[1+\mathrm{Re}(\beta_{k+1}^\ell a_{k+1} R_\ell \circ U_{k+1}^\ell)]\right)$. Recall that $\ell \ge 4 j_k$; hence, the harmonics $h_p, n_1 \le p \le n_k$, are the same for the polynomials φ_{k+1} and φ_k. Also, the polynomial φ_{k+1} has the required homogeneous expansion. Since $R_\ell \to 0$ weakly in $L^2(S_d)$, we have

$$\|\varphi_{k+1}(\ell) - \varphi_k[1+\mathrm{Re}(\beta_{k+1}^\ell a_{k+1} R_\ell \circ U_{k+1}^\ell)]\|_{C(S_d)} \to 0 \quad \text{as } \ell \to \infty$$

by (2.1). Thus, for all sufficiently large $\ell \in \mathbb{N}$, we have $\varphi_{k+1}(\ell) > 0$ and

$$\int_{S_d} \varphi_{k+1}^{\frac{1}{2}}(\ell) \, d\sigma_d \le \left(1 - \frac{\delta^2 |a_{k+1}|^2}{32} \right) \int_{S_d} \varphi_k^{\frac{1}{2}} \, d\sigma_d. \tag{2.5}$$

Finally, consider the polynomials h_p from the homogeneous expansion (2.2). Put

$$t_n = \frac{1}{2} + \sum_{p=1}^{n} h_p.$$

By (2.1), we have

$$\left\| \Pr\left[\beta_{k+1}^\ell a_{k+1} \bar{t}_n R_\ell \circ U_{k+1}^\ell \right] - \beta_{k+1}^\ell a_{k+1} \bar{t}_n R_\ell \circ U_{k+1}^\ell \right\|_{C(S_d)} \le |a_{k+1}| \tag{2.6}$$

for $n = 1, 2, \ldots, n_k$ provided that the index $\ell \in \mathbb{N}$ is sufficiently large.

Fix an index $\ell \in \mathbb{N}$ such that conditions (2.5) and (2.6) hold. Set $j_{k+1} = \ell$, $\beta_{k+1} = \beta_{k+1}^\ell$, $U_{j_{k+1}} = U_{k+1}^\ell$ and $\varphi_{k+1} = \varphi_{k+1}^\ell$. To simplify notation, in what follows, we write $R_{j_{k+1}}$ in the place of $\beta_{k+1} R_{j_{k+1}} \circ U_{j_{k+1}}$.

The spectral properties of the polynomials φ_k guarantee that the probability measures $\varphi_k \sigma_d$ converge weakly* to a probability measure $\pi = \pi(R, a, J) = \pi(R, a, J, \beta, U)$. So, we obtain a series with the following partial sums:

$$s_n = s_n[\pi] = 1 + \sum_{n_1 \le p \le n} (\overline{h}_p + h_p).$$

2.2.2. *A subsequence of partial sums converges to zero*

Fix a point $\zeta \in S_d$. Put $(\varphi_\zeta)_k(\lambda) = \varphi_k(\lambda\zeta)$, $\lambda \in \mathbb{T}$. Then the sequence $(\varphi_\zeta)_k m$ converges weakly* to a probability measure. Let π_ζ denote that limit. Note that the Fourier spectrum of π_ζ is contained in that of π. Hence, applying the inequality $j_{k+1}/j_k \ge 4$, we obtain $\widehat{\pi}_\zeta(j) = 0$ for

$n_k < |j| < 2n_k$. Thus, by Lemma 2.1(i),

$$(\varphi_k)_\zeta(\lambda) = s_{n_k}[\pi_\zeta](\lambda) \to \mathcal{D}\pi_\zeta(\lambda) \quad \text{for } m\text{-a.e. } \lambda \in \mathbb{T}.$$

Hence, the limit $\lim_{k\to\infty} \varphi_k(\zeta)$ exists for σ_d-a.e. $\zeta \in S_d$. Since $a \notin \ell^2$, we obtain $\varphi_k \to 0$ in $L^{\frac{1}{2}}(S_d)$ by (2.5). Therefore, $s_{n_k} = \varphi_k \to 0$ σ_d-a.e.

2.2.3. Convergence of the partial sums

Note that Lemma 2.1(ii) is applied only in this part of the proof. Namely, given a point $\zeta \in S_d$, consider the sequence

$$t_n(\zeta) = t_n[\pi](\zeta) = \frac{1}{2} + \sum_{n_1 \le p \le n} h_p(\zeta).$$

Lemma 2.1(ii) guarantees that the limit $\lim_{k\to\infty} t_{n_k}(\lambda\zeta)$ exists for m-a.e. $\lambda \in \mathbb{T}$. Therefore, the limit $\lim_{k\to\infty} t_{n_k}$ exists σ_d-a.e.

So, fix a point $\zeta \in S_d$ and assume that the subsequence $t_{n_k}(\zeta)$ converges. Below we show that the limit $\lim_{n\to\infty} t_n(\zeta)$ exists.

Note that $|t_{n_k}(\zeta)| \le M = M(\zeta) < \infty$ for all $k \in \mathbb{N}$. Put $G = 3M + 1$. By induction on k, below we verify the following property:

$$|t_n(\zeta)| \le G \quad \text{for all } 0 \le n \le n_k. \tag{2.7}$$

We have $t_{n_1} = (1 + a_{j_1} R_{j_1})/2$ and $t_n = 1/2$ for all $0 \le n < n_1$. Hence, the above estimate holds for $k = 1$. So, assuming that (2.7) holds for $k \in \mathbb{N}$, below we obtain the required property for $k + 1$.

Let $j_{k+1} - n_k \le n < j_{k+1}$. Consider the homogeneous expansion

$$t_{n_{k+1}} - t_n = \sum_{n < m \le n_{k+1}} h_m, \quad h_m \in H(m, 0).$$

Then h_m, $h_m \ne \mathbf{0}$, has the following form:

$$\frac{a_{k+1}}{2} R_{j_{k+1}} h_p,\ n_1 \le p \le n_k, \quad \text{or} \quad \frac{a_{k+1}}{2} R_{j_{k+1}}$$
$$\text{or} \quad \Pr\left(\frac{a_{k+1}}{2} R_{j_{k+1}} \overline{h}_p\right), \quad n_1 \le p \le j_{k+1} - n - 1.$$

Therefore,

$$\begin{aligned} t_{n_{k+1}} - t_n &= \Pr\left(\frac{a_{k+1}}{2} R_{j_{k+1}} [t_{n_k} + \overline{t}_{j_{k+1}-n-1}]\right) \\ &= \frac{a_{k+1}}{2} R_{j_{k+1}} t_{n_k} + \Pr\left(\frac{a_{k+1}}{2} R_{j_{k+1}} \overline{t}_{j_{k+1}-n-1}\right). \end{aligned}$$

Estimate (2.6) guarantees that

$$\left|\Pr\left(\frac{a_{k+1}}{2} R_{j_{k+1}} \overline{t}_{j_{k+1}-n-1}\right)\right| \le \frac{|a_{k+1}|}{2} |R_{j_{k+1}}(\zeta)||t_{j_{k+1}-n-1}(\zeta)| + \frac{|a_{k+1}|}{2}.$$

Thus, applying (2.7), we obtain

$$|t_n(\zeta) - t_{n_{k+1}}(\zeta)| \le \frac{|a_{k+1}|}{2}(M + G + 1) \quad \text{for } j_{k+1} - n_k \le n < j_{k+1}, \tag{2.8}$$

since $|t_{n_k}(\zeta)| \le M$.

Now, assume that $j_{k+1} \le n \le n_{k+1}$. Then

$$\begin{aligned} t_n - t_{n_k} &= \Pr\left(\frac{a_{k+1}}{2} R_{j_{k+1}}[t_{n-j_{k+1}} + \bar{t}_{n_k}]\right) \\ &= \frac{a_{k+1}}{2} R_{j_{k+1}} t_{n-j_{k+1}} + \Pr\left(\frac{a_{k+1}}{2} R_{j_{k+1}} \bar{t}_{n_k}\right). \end{aligned}$$

Hence, estimates (2.6) and (2.7) guarantee that

$$|t_n(\zeta) - t_{n_k}(\zeta)| \le \frac{|a_{k+1}|}{2}(G + M + 1) \quad \text{for } j_{k+1} \le n \le n_{k+1},$$

since $|\bar{t}_{n_k}(\zeta)| \le M$. Also, we have $t_n = t_{n_k}$ for $n_k < n < j_{k+1} - n_k$. Therefore,

$$|t_n(\zeta) - t_{n_k}(\zeta)| \le \frac{|a_{k+1}|}{2}(M + G + 1) \quad \text{for } n_k < n < j_{k+1} - n_k \\ \text{and for } j_{k+1} \le n \le n_{k+1}. \tag{2.9}$$

Recall that $|a_{k+1}| < 1$, $|t_{n_k}(\zeta)| \le M$ and $|t_{n_{k+1}}(\zeta)| \le M$. Hence, by (2.8) and (2.9), we obtain

$$|t_n(\zeta)| \le M + \frac{|a_{k+1}|}{2}(M + G + 1) \le G \quad \text{for all } n_k < n \le n_{k+1}.$$

In other words, inequality (2.7) holds for $k + 1$ in the place of k. Now, the induction construction works.

Since $a_{k+1} \to 0$, we have $\lim_{n\to\infty} t_n(\zeta) = \lim_{k\to\infty} t_{n_k}(\zeta)$ by (2.8) and (2.9). So, the limit $\lim_{n\to\infty} t_n$ exists σ_d-a.e. Note that $s_n = 2\mathrm{Re} t_n$. Hence, the limit $\lim_{n\to\infty} s_n(\zeta)$ exists for σ_d-a.e. $\zeta \in S_d$. Recall that $s_{n_k} \to 0$ σ_d-a.e., therefore, $s_n \to 0$ σ_d-a.e. The proof of Theorem 1.1 is completed.

2.2.4. Final comments

Let $E \subset \mathbb{Z}_+$ contain arbitrarily long intervals. Then, applying the above argument, we may additionally guarantee that $h_j = 0$ for all $j \in \mathbb{Z}_+ \setminus E$.

As mentioned above, the proof of Theorem 1.1 is based on the pluriharmonic Riesz product construction from [1]. The crucial additional restriction is (2.6).

3. Generalized pluriharmonic Riesz products

To prove Theorems 1.2 and 1.3, we replace the homogeneous Ryll–Wojtaszczyk polynomials by appropriate polynomials with lacunary spectrum. The corresponding modification of the Riesz construction on the circle group is called a generalized product. We will distinguish L^2- and L^∞-generalized Riesz pairs and products on the sphere S_d.

3.1. L^2-generalized Riesz products

3.1.1. L^2-generalized Riesz pairs

Assume that, for any $j, L \in \mathbb{N}$, we are given a number $\Gamma = \Gamma(j, L; d) \in \mathbb{N}$ and holomorphic homogeneous polynomials $W_\gamma = W_\gamma(j, L; d)$, $\gamma = 1, 2, \ldots, \Gamma$, such that

- $\deg W_\gamma \ge j$;
- $\left|\deg W_{\gamma_1} - \deg W_{\gamma_2}\right| \ge L$ for $\gamma_1 \ne \gamma_2$ (lacunas in the spectrum);
- $\left|\sum_{\gamma=1}^{\Gamma} W_\gamma(\zeta)\right| \le 1$ for all $\zeta \in S_d$;
- $\sum_{\gamma=1}^{\Gamma} \left\| W_\gamma \right\|^2_{L^2(S_d)} \ge \delta$ for a constant $\delta = \delta(d) > 0$.

Put

$$R(j, L) = \sum_{\gamma=1}^{\Gamma} W_\gamma(j, L)$$

and define $R = \{R(j, L)\}_{j,L}$. Also, fix a coefficient sequence $a = \{a_k\}_{k=1}^{\infty} \subset \mathbb{C}$, $0 < |a_k| < 1$. Then (R, a) is called an L^2*-generalized Riesz pair*.

Clearly, the spectral lacunas assumption for the polynomials $R(j, L)$ becomes trivial if $\Gamma(j, L) = 1$. In other words, a Riesz pair is an L^2-generalized one. Other examples of L^2-generalized Riesz pairs are provided by Lemma from [2]; see also [3, Lemma 2.1].

3.1.2. Induction construction

Fix an L^2-generalized Riesz pair (R, a). By induction, we construct an index set $J = \{j_k\}_{k=1}^{\infty}$ and a sequence of pluriharmonic polynomials φ_k such that $\varphi_k > 0$ on the sphere S_d.

First, fix $j_1 \in \mathbb{N}$ and put $\varphi_1 = 1 + \mathrm{Re}\,[a_1 R(j_1, 1)]$.

Step $k + 1$. By the induction hypothesis, we are given a pluriharmonic polynomial φ_k, $\varphi_k > 0$ on S_d. Put $L_{k+1} = 2 \deg \varphi_k + 2$ and define

$$\varphi_{k+1}(\ell) = \mathrm{Pr}\left[\varphi_k \left(1 + \mathrm{Re}\left[a_{k+1} R(\ell, L_{k+1})\right]\right)\right]. \tag{3.1}$$

Property (2.1) guarantees that $\varphi_{k+1}(\ell) > 0$ on S_d for all sufficiently large ℓ. Next, if ℓ is large enough, then the degrees of the polynomials in the homogeneous expansion of the difference $\varphi_{k+1}(\ell) - \varphi_k$ are strictly larger than $\deg \varphi_k$.

Fix so large number ℓ that the above restrictions hold. Put $j_{k+1} = \ell$ and $\varphi_{k+1} = \varphi_{k+1}(j_{k+1})$.

Note that the choice of L_{k+1}, the size of lacuna, guarantees the key property in the definition of a Riesz product. Namely, let c_m, $m = 1, 2, \ldots$, denote the degrees of the non-trivial polynomials in the homogeneous expansion of φ_{k+1}. Then any $r \in \mathbb{Z}$ has at most one representation of the following type:

$$r = \sum_{c_{m_1} \neq c_{m_2}} \pm c_m.$$

The above restrictions guarantee that the sequence of probability measures $\varphi_k \sigma_d$ weakly* converges in the space of Borel measures on S_d. The limit probability measure $\pi = \pi(R, a, J)$ is called an L^2*-generalized pluriharmonic Riesz product*.

To prove the following assertion, it suffices to repeat the model arguments from Sections 2.2.1 and 2.2.2; cf. [2], where the proof is somewhat different.

Proposition 3.1. *Let (R, a) be an L^2-generalized Riesz pair such that $a \notin \ell^2$. Then, for every sufficiently lacunary index set $J = \{j_k\}_{k=1}^{\infty} \subset \mathbb{N}$, there exists a sign sequence $\beta = \{\beta_k\}_{k=1}^{\infty}$, $\beta_k \in \{\pm 1\}$, and a sequence of unitary operators $U = \{U_k\}_{k=1}^{\infty}$ such that*

(i) *$\varphi_k \to 0$ σ_d-a.e.,*
(ii) *$\pi(R \circ U, \beta a, J) \perp \sigma_d$.*

3.1.3. Comments

It is necessary to explain certain details of Proposition 3.1. By definition,

$$\varphi_{k+1} = \mathrm{Pr}\left[\varphi_k \left(1 + \mathrm{Re}\left[\beta_{k+1} a_{k+1} R(j_{k+1}, L_{k+1}) \circ U_{j_{k+1}}\right]\right)\right],$$

$R \circ U = \{R(j_k, L_k) \circ U_k\}_{k=1}^{\infty}$ and $\beta a = \{\beta_k a_k\}_{k=1}^{\infty}$. To construct the measure $\pi(R \circ U, \beta a, J)$, we use arguments from Section 2.2.1, hence, the index sequence $J = \{j_k\}_{k=1}^{\infty}$ is obtained by induction. So, in applications, Proposition 3.1 is used as follows: on the induction step $k+1$, put

$$\varphi_{k+1}(\ell) = \Pr\left[\varphi_k\left(1 + \operatorname{Re}\left[\beta_{k+1}^{\ell} a_{k+1} R(\ell, L_{k+1}) \circ U_{k+1}^{\ell}\right]\right)\right], \quad \ell \in \mathbb{N}.$$

For all sufficiently large $\ell \in \mathbb{N}$, we have $\varphi_{k+1}(\ell) > 0$ on S_d; also, there exist unitary operators U_{k+1}^{ℓ} and signs β_{k+1}^{ℓ} such that (2.5) holds. Finally, we impose additional restrictions that hold for all sufficiently large $\ell \in \mathbb{N}$. In the model case, such a restriction is inequality (2.6). Selecting appropriate ℓ, put $j_{k+1} = \ell$. The result of the induction construction is a singular measure $\pi(R \circ U, \beta a, J)$. In what follows, we often omit the auxiliary sequences U and β.

3.2. L^{∞}-generalized Riesz pairs and products

Assume that, for any $j, L \in \mathbb{N}$, we are given holomorphic homogeneous polynomials $W_{\gamma} = W_{\gamma}(j, L; d)$, $\gamma = 1, 2, \dots, \Gamma = \Gamma(d)$, such that

- $\deg W_1 \geq j$;
- $\deg W_{\gamma+1} - \deg W_{\gamma} \geq L$ for $\gamma = 1, 2, \dots, \Gamma - 1$;
- $\sum_{\gamma=1}^{\Gamma} |W_{\gamma}(\zeta)| \leq 1$ for all $\zeta \in S_d$;
- $\sum_{\gamma=1}^{\Gamma} \left|W_{\gamma}(\zeta)\right|^2 \geq \delta > 0$ for all $\zeta \in S_d$, where δ is a constant that depends only on d.

The above polynomials exist by Lemma 2.1 from [3]. Note that the constant Γ does not depend on j and L in this case.

Put $R(j, L) = \sum_{\gamma} W_{\gamma}(j, L)$ and $R = \{R(j, L)\}_{j,L}$. It is worth mentioning that the polynomials $R(j, L)$ could not be homogeneous, that is, $R(j, L)$ could not be replaced by Ryll–Wojtaszczyk polynomials. Also, fix a coefficient sequence $a = \{a_k\}_{k=1}^{\infty} \subset \mathbb{C}$, $|a_k| < 1$. Then (R, a) is called an L^{∞}-generalized Riesz pair.

Note that an L^{∞}-generalized Riesz pair is an L^2-generalized one. Therefore, applying the construction from Section 3.1.2, we define the L^{∞}-generalized pluriharmonic Riesz products.

More stringent restrictions imposed on the L^{∞}-generalized Riesz pairs guarantee more predictable behavior of the L^{∞}-generalized products in comparison with their L^2-counterparts. In particular, the auxiliary sequences U and β are not used in the following assertion.

Proposition 3.2 (*[3, Theorem 2(ii)]*). *Let (R, a) be an L^{∞}-generalized Riesz pair such that $a \notin \ell^2$. Then, for all sufficiently lacunary index sets $J \subset \mathbb{N}$, the slice-measure $\pi_{\zeta}(R, J, a)$ and the Lebesgue measure on the circle $\mathbb{T}$ are mutually singular for all points $\zeta \in S$. In particular, $\pi(R, J, a) \perp \sigma_d$.*

Comments to the above proposition are similar to those given in Section 3.1.3: the index set $J \subset \mathbb{N}$ is constructed by induction, on step $k+1$ we impose certain restrictions that hold for all sufficiently large $\ell \in \mathbb{N}$ and we put $j_{k+1} = \ell$ for such an ℓ. So, in applications, on step $k+1$ we are allowed to add new properties that hold for all sufficiently large $\ell \in \mathbb{N}$.

4. Proof of Theorem 1.2

To construct the required null-series, we apply Proposition 3.2. Also, we use the argument applied in Section 2.2.3 as a model. So, fix an L^{∞}-generalized Riesz pair (R, a) such that $a \notin \ell^2$

and $a_k \to 0$. Recall that

$$R(j, L) = \sum_{\gamma=1}^{\Gamma} W_\gamma(j, L).$$

4.1. Induction construction

Let $k \in \mathbb{N}$, $\varphi_k > 0$ on the sphere S_d and the polynomial φ_k has the following homogeneous expansion:

$$\varphi_k = \sum_{p=1}^{n_k} \overline{h}_p + 1 + \sum_{p=1}^{n_k} h_p,$$

where $h_p \in H(p, 0)$ and $n_k = \deg \varphi_k$. As in the model argument, put

$$s_n = 1 + \sum_{p=1}^{n} (\overline{h}_p + h_p),$$

$$t_n = \frac{1}{2} + \sum_{p=1}^{n} h_p$$

for $n = 0, 1, \dots, n_k$.

Step $k+1$. Below we assume that $\ell \in \mathbb{N}$ is sufficiently large. Namely, the polynomial

$$\varphi_{k+1}(\ell) = \Pr\left(\varphi_k[1 + \operatorname{Re}(a_{k+1} R(\ell, L_{k+1}))]\right)$$

has the properties that guarantee the conclusion of Proposition 3.2. Also, we impose additional restrictions that hold for all $\ell \in \mathbb{N}$ large enough.

Recall that

$$\left\| \sum_{\gamma=1}^{r} W_\gamma(\ell, L_{k+1}) \right\|_{C(S_d)} \le 1 \quad \text{for } r = 1, 2, \dots, \Gamma.$$

Thus, property (2.1) guarantees that

$$\left\| \Pr\left[a_{k+1} \sum_{\gamma=1}^{r} W_\gamma(\ell, L_{k+1}) \varphi_k \right] - a_{k+1} \sum_{\gamma=1}^{r} W_\gamma(\ell, L_{k+1}) \varphi_k \right\|_{C(S_d)} \le |a_{k+1}| \tag{4.1}$$

for $r = 1, 2, \dots, \Gamma$ and for all sufficiently large $\ell \in \mathbb{N}$. Analogously, we have

$$\left\| \Pr\left[a_{k+1} \sum_{\gamma=r+1}^{\Gamma} W_\gamma(\ell, L_{k+1}) \varphi_k \right] - a_{k+1} \sum_{\gamma=r+1}^{\Gamma} W_\gamma(\ell, L_{k+1}) \varphi_k \right\|_{C(S_d)} \le |a_{k+1}| \tag{4.2}$$

for $r = 1, 2, \dots, \Gamma - 1$ and for all sufficiently large $\ell \in \mathbb{N}$.

Next, using the inequality $\|W_\gamma(\ell, L_{k+1})\|_{C(S_d)} \le 1$ and applying property (2.1), we obtain

$$\left\| \Pr\left[a_{k+1} W_\gamma(\ell, L_{k+1}) \overline{t}_n \right] - a_{k+1} W_\gamma(\ell, L_{k+1}) \overline{t}_n \right\|_{C(S_d)} \le |a_{k+1}| \tag{4.3}$$

for $\gamma = 1, 2, \dots, \Gamma$, $n = 1, 2, \dots, n_k$ and for all sufficiently large $\ell \in \mathbb{N}$. Note that condition (4.3) is a direct analogue of (2.6).

Fix so large number ℓ that all above restrictions are fulfilled. Put $j_{k+1} = \ell$ and $\varphi_{k+1} = \varphi_{k+1}(j_{k+1})$. Now, the induction construction proceeds.

4.2. A subsequence of partial sums converges

Fix a point $\xi \in S_d$. By Proposition 3.2, the slice-measure π_ξ is singular with respect to Lebesgue measure m. Note that $\widehat{\pi}_\xi(j) = 0$ for $n_k < |j| < 2n_k$. Hence, applying Lemma 2.1(i), we obtain

$$\varphi_k(\lambda\xi) = s_{n_k}(\lambda\xi) \to 0 \quad \text{for } m\text{-a.e. } \lambda \in \mathbb{T}.$$

Also, the limit $\lim_{k\to\infty} t_{n_k}(\lambda\xi)$ exists for m-a.e. $\lambda \in \mathbb{T}$ by Lemma 2.1(ii).

So, fix a point $\zeta \in S_d$ such that the subsequence $t_{n_k}(\zeta)$ converges. Below we prove that the sequence $t_n(\zeta)$ also converges.

4.3. Convergence of the partial sums

4.3.1. An auxiliary condition

We have $|t_{n_k}(\zeta)| \le M = M(\zeta) < \infty$ for all $k \in \mathbb{N}$. Put $G = 5M + 2$ and consider the following property:

$$|t_n(\zeta)| \le G \quad \text{for all } 0 \le n \le n_k. \tag{4.4}$$

If $0 \le n \le n_1$, then

$$t_n = \frac{1}{2} + \sum_{\gamma=1}^{A} \frac{a_1}{2} W_\gamma(j_1, 1)$$

for certain A, $0 \le A \le \Gamma$. Hence, (4.4) holds for $k = 1$. Now, we argue by induction. So, assume that (4.4) holds with some $k \in \mathbb{N}$.

To simplify notation, in what follows, we write W_γ in the place of $W_\gamma(j_{k+1}, L_{k+1})$, $1 \le \gamma \le \Gamma$.

Recall that $n_k = \deg \varphi_k$ and $n_{k+1} = \deg \varphi_{k+1}$. Let $n \in \mathbb{N}$, $n_k < n \le n_{k+1}$. If $n_k < n < \deg W_1 - n_k$, then $t_n = t_{n_k}$. So, below we assume that

$$\deg W_1 - n_k < n \le n_{k+1}.$$

Put $r = r(n) = \max\{\gamma \le \Gamma : n \ge \deg W_\gamma - n_k\}$. Clearly, $1 \le r \le \Gamma$. Set $\omega = \deg W_r$.

If $1 \le r < \Gamma$, then $\omega - n_k \le n < \deg W_{r+1} - n_k$ and $t_n = t_{\omega+n_k}$ for $\omega + n_k < n < \deg W_{r+1} - n_k$. If $r = \Gamma$, then $\omega - n_k \le n \le n_{k+1} = \omega + n_k$. Hence, without loss of generality, we may assume that

$$\omega - n_k \le n \le \omega + n_k.$$

4.3.2. Let $\omega - n_k \le n < \omega$

Then

$$\begin{aligned} t_{n_{k+1}} - t_n &= \Pr\left[\frac{a_{k+1}}{2} \sum_{\gamma=r+1}^{\Gamma} W_\gamma \varphi_k + \frac{a_{k+1}}{2} W_r t_{n_k} + \frac{a_{k+1}}{2} W_r \bar{t}_{\omega-n-1}\right] \\ &= \Pr(\Sigma_1 + \Sigma_2 + \Sigma_3), \end{aligned}$$

where Σ_1, Σ_2 and Σ_3 are defined by the last identity; $\Sigma_1 = 0$ for $r = \Gamma$.

First,

$$\|\mathrm{Pr}\Sigma_1 - \Sigma_1\|_{C(S_d)} \le \frac{|a_{k+1}|}{2}$$

by (4.2). Also, we have $|\varphi_k| \le 2|t_{n_k}|$, $|t_{n_k}(\zeta)| \le M$ and

$$\left\| \sum_{\gamma=r+1}^{\Gamma} W_\gamma \right\|_{C(S_d)} \le 1.$$

Therefore,

$$|\mathrm{Pr}\Sigma_1(\zeta)| \le \frac{|a_{k+1}|}{2} + |\Sigma_1(\zeta)| \le \frac{|a_{k+1}|}{2}(2M+1).$$

Second,

$$|\mathrm{Pr}\Sigma_2(\zeta)| = |\Sigma_2(\zeta)| \le \frac{|a_{k+1}|}{2} M$$

because $|t_{n_k}(\zeta)| \le M$ and $\|W_r\|_{C(S_d)} \le 1$.

Third, applying (4.3), we obtain

$$|\mathrm{Pr}\Sigma_3(\zeta)| \le \frac{|a_{k+1}|}{2} + |\Sigma_3(\zeta)| \le \frac{|a_{k+1}|}{2}(G+1)$$

by (4.4) with $n = n_k$.

In sum, we have

$$|t_{n_{k+1}} - t_n| \le \frac{|a_{k+1}|}{2}(G+3M+2) \quad \text{for } \omega - n_k \le n < \omega. \tag{4.5}$$

4.3.3. Let $\omega \le n \le n_{k+1}$

Then

$$\begin{aligned} t_n - t_{n_k} &= \mathrm{Pr}\left[\frac{a_{k+1}}{2} \sum_{\gamma=1}^{r-1} W_\gamma \varphi_k + \frac{a_{k+1}}{2}\, W_r t_{n-\omega} + \frac{a_{k+1}}{2}\, W_r \bar{t}_{n_k} \right] \\ &= \mathrm{Pr}(\Sigma_4 + \Sigma_5 + \Sigma_6), \end{aligned}$$

where Σ_4, Σ_5 and Σ_6 are defined by the last identity; $\Sigma_4 = 0$ for $r = 1$. Now, we argue as in Section 4.3.2. So,

$$\begin{aligned} |\mathrm{Pr}\Sigma_4(\zeta)| &\le \frac{|a_{k+1}|}{2} + |\Sigma_4(\zeta)| \le \frac{|a_{k+1}|}{2}(2M+1), \\ |\mathrm{Pr}\Sigma_5(\zeta)| &= |\Sigma_5(\zeta)| \le \frac{|a_{k+1}|}{2} G, \\ |\mathrm{Pr}\Sigma_6(\zeta)| &\le \frac{|a_{k+1}|}{2} + |\Sigma_6(\zeta)| \le \frac{|a_{k+1}|}{2}(M+1) \end{aligned}$$

by (4.1), (4.3) and (4.4). In sum, we obtain

$$|t_n(\zeta) - t_{n_k}(\zeta)| \le \frac{|a_{k+1}|}{2}(G+3M+2) \quad \text{for } \omega \le n \le n_{k+1}. \tag{4.6}$$

4.3.4. Final step of the proof of Theorem 1.2

We have $|a_{k+1}| < 1$, $|t_{n_k}(\zeta)| \le M$ and $|t_{n_{k+1}}(\zeta)| \le M$, therefore,

$$|t_n(\zeta)| \le M + \frac{|a_{k+1}|}{2}(3M + G + 2) \le G \quad \text{for all } n_k < n \le n_{k+1}$$

by (4.5) and (4.6). In other words, inequality (4.4) holds for $k+1$ in the place of k. So, the induction construction proceeds.

Recall that $a_{k+1} \to 0$. Thus, we obtain $\lim_{n\to\infty} t_n(\zeta) = \lim_{k\to\infty} t_{n_k}(\zeta)$ by (4.5) and (4.6). So, as in the proof of Theorem 1.1, we conclude that $s_n \to 0$ σ_d-a.e. The proof of Theorem 1.2 is completed.

5. Proof of Theorem 1.3

We need the polynomials provided by the following lemma.

Lemma 5.1 (*See [2, Lemma]*). *Let $d \ge 2$ and let $j, L \in \mathbb{N}$. Then there exist homogeneous holomorphic polynomials $W_\gamma = W_\gamma(j, L)$ of $z_1, z_2, \dots, z_d$, $\gamma = 1, 2, \dots, \Gamma(j, L)$, such that*

$$\deg W_1 \ge j; \tag{5.1}$$

$$\deg W_{\gamma+1} - \deg W_\gamma \ge L \quad \textit{for } \gamma = 1, 2, \dots, \Gamma(j, L) - 1; \tag{5.2}$$

$$\sum_\gamma |W_\gamma(\zeta)| \le 1 \quad \textit{for all } \zeta \in S_d; \tag{5.3}$$

$$\sum_\gamma \|W_\gamma(\zeta)\|^2_{L^2(S_d)} \ge \delta; \tag{5.4}$$

$$\|W_\gamma(\zeta)\|^2_{L^2(S_d)} \le \frac{C}{\deg W_\gamma} \tag{5.5}$$

for constants $\delta = \delta(d) > 0$ and $C = C(d) > 0$.

Note that property (5.3) is verified, but not explicitly formulated in [2].

Let $W_\gamma(j, L)$ be the polynomials provided by Lemma 5.1 and let $a \notin \ell^2$, $a_k \to 0$. Put $R(j, L) = \sum_\gamma W_\gamma(j, L)$. Then (R, a) is a Riesz pair by (5.1)–(5.4). To construct a null-series, we combine Proposition 3.1 and the arguments from the proof of Theorem 1.2. Namely, we apply Proposition 3.1 as described in Section 3.1.3. So, on step $k+1$ of the induction construction, we consider the following polynomials:

$$\varphi_{k+1}(\ell) = \Pr\left[\varphi_k\left(1 + \operatorname{Re}\left[\beta^\ell_{k+1} a_{k+1} R(\ell, L_{k+1}) \circ U^\ell_{k+1}\right]\right)\right], \quad \ell \in \mathbb{N}.$$

Recall that it is allowed to add restrictions that hold for all sufficiently large $\ell \in \mathbb{N}$. As usual, we write $W_\gamma(j, L)$ in the place of $\beta^\ell_{k+1} W_\gamma(j, L) \circ U^\ell_{k+1}$. So, we may repeat the arguments given in Section 4. First, estimates (4.1)–(4.3) hold for all sufficiently large $\ell \in \mathbb{N}$ by (2.1) and (5.3). Second, we argue as in Section 2.2.2. Clearly, this step differs from the proof of Theorem 1.2: we apply Proposition 3.1 in the place of Proposition 3.2. Finally, we repeat word by the word, the arguments from Section 4.3. As a result, we obtain a null-series

$$1 + \sum_{p=1}^{\infty} (\overline{h}_p + h_p), \quad h_p \in H(p, 0). \tag{5.6}$$

It remains to ensure that $\|h_p\|^2_{L^2(S_d)} \le Cp^{-1}$. So, consider the homogeneous expansion

$$\varphi_k = \sum_{p=1}^{n_k} \overline{h}_{p,k} + 1 + \sum_{p=1}^{n_k} h_{p,k}, \quad h_{p,k} \in H(p,0).$$

By induction on k, we ensure that

$$\|h_{p,k}\|_{C(S_d)} \le 1 \quad \text{for } p = 1, 2, \dots, n_k. \tag{5.7}$$

Clearly, the above property holds for $k = 1$. So, assume as an induction hypothesis, that (5.7) holds for some $k \in \mathbb{N}$.

On step $k+1$, we impose an additional restriction. Let $h_{m,k+1}(\ell)$ denote a non-zero $H(m,0)$-projection of $\varphi_{k+1}(\ell) - \varphi_k$. Then $h_{m,k+1}(\ell)$ has the following form:

$$\frac{a_{k+1}}{2}\mathrm{Pr}[W_r \overline{h}_{p,k}] \quad \text{or} \quad \frac{a_{k+1}}{2} W_r \quad \text{or} \quad \frac{a_{k+1}}{2} W_r h_{p,k}.$$

To verify property (5.7) with $k+1$ in the place of k, it suffices to consider the case, where $h_{m,k+1}(\ell) = a_{k+1}\mathrm{Pr}[W_r \overline{h}_{p,k}]/2$. Property (2.1) guarantees that

$$\|h_{m,k+1}(\ell) - a_{k+1}[W_r \overline{h}_{p,k}]/2\|_{C(S_d)} \le 1/2 \tag{5.8}$$

for all $r = 1, 2, \dots, \Gamma(\ell, L_{k+1})$, $p = 1, 2, \dots, n_k$ and for all sufficiently large $\ell \in \mathbb{N}$. Now, set $j_{k+1} = \ell$ and $\varphi_{k+1} = \varphi_{k+1}(j_{k+1})$, where ℓ is so large that all additional restrictions hold.

Applying estimates (5.7) and (5.8), we obtain

$$2\|h_{m,k+1}\|_{C(S_d)} \le 1 + \|W_r h_{p,k}\|_{C(S_d)} \le 2.$$

So, (5.7) holds with $k+1$ in the place of k. Hence, the induction construction works. Also, note that

$$\|h_{m,k+1}\|^2_{L^2(S_d)} \le \|W_r h_{p,k}\|^2_{L^2(S_d)} \le \|W_r\|^2_{L^2(S_d)} \|h_{p,k}\|^2_{C(S_d)} \le C\,(\deg W_r)^{-1};$$
$$\deg(W_r h_{p,k}) \le 2 \deg W_r.$$

The above estimates guarantee that we obtain a null-series (5.6) such that $\|h_p\|^2_{L^2(S_d)} \le Cp^{-1}$. The proof of Theorem 1.3 is completed.

Acknowledgments

The second author was partially supported by RFBR (grant No. 11-01-00526-a).

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